Vector Potential and Magnetic Field of Current-carrying Finite Elliptic Arc Segment in Analytical Form

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Herrn Professor Dr. Fritz Bopp zur Vollendung seines 75. Lebenjahres gewidmet

Analytical expressions for the components of the vector potential and magnetic field of a current-carrying elliptic arc filament of arbitrary length are derived. All the expressions developed consist of only known functions such as Jacobian elliptic functions, complete and incomplete elliptic integrals of the first, second and third kind, and thus permit a compact time-saving efficient computation algorithm.

Introduction

In many problems involving physical and technological applications of large-scale magnetic fields it is sufficient to assume the conductor cross-sectional area of the field coil to be negligible. In general, this assumption leads to a filament approximation discussed in detail in Part I of a series of papers on magnetic field computations in a closed form [1]. Analytical expressions consisting of Jacobian elliptic functions, complete and incomplete ellitic integrals of the first, second and third kind were derived for the components of the vector potential and magnetic field of a finite circular arc segment of arbitrary angular length.

In the present paper we generalize our results of Part I to current-carrying elliptic arc filaments of arbitrary azimuthal lengths. This case, to our knowledge, has not yet been treated analytically in the literature although magnetic field coils of elliptic form are interesting for the manipulation of particle beams in high energy physics, for asymmetric and special field profile magnets, for correction coils of highly homogeneous special purpose magnets, etc. Computation methods known up to date for calculating the vector potential and magnetic field in such cases use a numerical integration of the basic equations employing circular, wedge or beam approximations [2].

Since a field coil of arbitrary spatial geometry may be made from finite elliptic arc and/or straight segments of the conductor, its magnetic field is given by the sum over the partial fields generated by each segment. A computer time-saving effect of the closed expressions over the usual method of numerical integration of the basic equations then is quite apparent.

When the elliptical arc filament is 2π long, that is for a closed elliptic loop, the expressions developed reduce to relatively simple ones. For zero eccentricity they also reduce to those derived in Part I, both for the circular arc filament and for the circular filament loop.

Basic Expressions

Starting from the law of Biot-Savart, the vector potential and magnetic induction due to an arbitrary current filament are given by

$$\bar{A}(\bar{r}) = J/4\,\pi\,\bar{\bar{A}}(\bar{r})$$
$$= J/4\,\pi \int_l \int_s \mathrm{d}\bar{l}\,|\bar{r} - \bar{r}'|^{-1}\,\delta(s)\,\mathrm{d}s\,, \qquad (1)$$

$$\bar{B}(\bar{r}) = \mu_0 J/4\,\pi\,\bar{\bar{H}}(\bar{r})$$
$$= \mu_0 J/4\,\pi \int_l \int_s [\mathrm{d}\bar{l}\,x\,(\bar{r} - \bar{r}')]\,|r - r'|^{-3}\,\delta(s)\,\mathrm{d}s\,, \qquad (2)$$

where $\mathrm{d}\bar{l}$ is a vector differential line element with a constant current J, $\bar{r}'$ and $\bar{r}$ are position vectors of the source and field point, respectively, and μ_0 is the free-space permeability. $\mathrm{d}s$ is a differential element
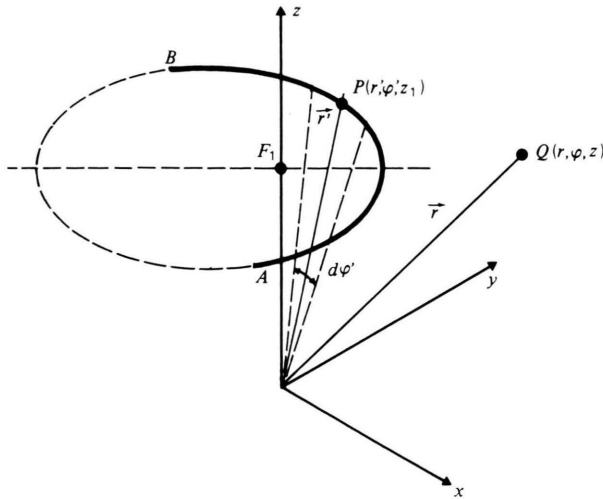
Fig. 1. Elliptic arc filament of finite length AB at spatial height z_1 and focus F_1 on z axis in local cylinder coordinates.

of cross-sectional area orthogonal to $d\bar{l}$, and $\delta(s)$ is a two-dimensional Dirac delta function.

For an elliptic arc filament in the local cylinder coordinates (Fig. 1) we have $\delta(s) = \delta(\varrho' - r')\,\delta(z' - z_1)$, where r' is its angle dependent radial ray, z_1 the spatial height of its plane parallel to the $z = 0$ ground plane, and ϱ', φ' and z' are the coordinates of the variable volume element.

The volume integrals in (1) and (2) may now be performed in three steps. In the first two, the integration over the cross-sectional area can be done easily due to the δ-functions. We first integrate over z' and then over ϱ' with φ' as parameter. In the third step, we integrate over Φ after transforming φ' to a new variable Φ differing from it by an additive constant only and hence not affecting the results of the integration.

Since $A_z = 0$, we get from (1) and (2) after integrating over the cross-sectional area

$$\hat{A}_j(\bar{r}) = \int_{\varphi_1}^{\varphi_2} d\Phi\, r'(\Phi)\, D^{-1}(\Phi) \begin{cases} -\sin\Phi \\ \cos\Phi \end{cases} \quad (j = r, \varphi), \quad (3)$$

$$\hat{H}_l(\bar{r}) = \int_{\varphi_1}^{\varphi_2} d\Phi\, r'(\Phi)\, D^{-3}(\Phi)$$

$$\begin{cases} -\gamma\cos\Phi \\ -\gamma\sin\Phi \\ r'(\Phi) - r\cos\Phi \end{cases} \quad (l = r, \varphi, z), \quad (4)$$

where the column terms correspond, from the top, to $j = r$, φ and $l = r$, φ, z components of the vector potential and magnetic field, respectively. $\bar{r}'(r', \varphi', z_1)$ and $\bar{r}(r, \varphi, z)$ are the coordinates of the source and field point with respect to the origin, and r' is a function of φ'. Also $\Phi_i = \varphi_i - \varphi$, where ($i = 1, 2$) and

$$D^2(\Phi) = \gamma^2 + r'^2(\Phi) + r^2 - 2\, r\, r'(\Phi)\cos\Phi,$$

$$\gamma = z_1 - z, \quad \Phi = \varphi' - \varphi,$$

$(\varphi_2 - \varphi_1)$ is the azimuthal length of the elliptic arc filament AB (Figure 1). Note that the positive x-axis corresponds to $\varphi = 0$. φ increases in counterclockwise direction.

From (3) and (4) a straight-forward but tedious and lengthy computation, exactly analogous to that in [3] [Appendix], shows that $\nabla^2\hat{\bar{A}} = 0$ and $\bar{\nabla} \times \bar{H} = 0$ in a current-free region. Only for a closed elliptic loop and $\varphi = 0$, these evaluations are slightly easier.

Method of Evaluation

In order to evaluate the expressions (3) and (4) in a closed form, we transform them into integrals over Jacobian elliptic functions. To this end we first introduce, as earlier, the system invariant angle transformation [1]

$$\Phi = \pi - 2\,\alpha. \quad (A)$$

so that $\alpha_i = 1/2\,(\pi + \varphi - \varphi_i)$. For convenience, we also introduce, as before, the symbolic representation

$$\left.\begin{matrix} \hat{A}_j(\bar{r}) \\ \hat{H}_l(\bar{r}) \end{matrix}\right\} = \sum_{i=1}^{2} (-1)^{i+1} \quad (5)$$

$$\cdot \begin{cases} \hat{A}_j(|\alpha_i|)\,(\delta_{jr} + \delta_{j\varphi}\operatorname{sgn}\alpha_i)\,(j = r, \varphi), \\ \hat{H}_l(|\alpha_i|)\,(\delta_{l\varphi} + \delta_{lm}\operatorname{sgn}\alpha_i)\,(l = r, \varphi, z; m = r, z), \end{cases}$$

where δ_{ij} is the Kronecker delta.

Using the polar equation for an ellipse with respect to its right focus F_1 as pole, we obtain

$$r'(\alpha) = r_1/(1 + n_3^2\sin^2\alpha) \quad (6)$$

with $n_3^2 = 2\varepsilon/(1 - \varepsilon)$ and $r_1 = \eta(1 + \varepsilon)$. 2η is the major axis of the ellipse and ε its numerical eccentricity.

From (3)–(6) we get after some algebra the following hyperelliptic integrals for the vector

potential and the magnetic field, α being positive definite:

$$\hat{A}_j(\alpha) = -\frac{k_0^2 a}{2r} \int_0^\alpha \frac{d\alpha}{(1 - n_1^2 \sin^2 \alpha)^{1/2} (1 - n_2^2 \sin^2 \alpha)^{1/2}} \begin{cases} \sin 2\alpha \\ \cos 2\alpha \end{cases} \quad (j = r, \varphi), \tag{7}$$

$$\hat{H}_l(\alpha) = \frac{k_0^2}{2ar} \int_0^\alpha \frac{d\alpha (1 + n_3^2 \sin^2 \alpha)^2}{(1 - n_1^2 \sin^2 \alpha)^{3/2} (1 - n_2^2 \sin^2 \alpha)^{3/2}} \begin{cases} \gamma \cos 2\alpha \\ -\gamma \sin 2\alpha \\ r \cos 2\alpha + \dfrac{r_1}{(1 + n_3^2 \sin^2 \alpha)} \end{cases} \quad (l = r, \varphi, z). \tag{8}$$

The dimensionless parameters n_i^2 with $(i = 1, 2)$ of the hyperelliptic integrals are defined through *

$$n_i^2 = 2 d^2 / [b \mp (b^2 - 4 d^2)^{1/2}], \tag{9a}$$

the upper (lower) sign corresponding to $i = 1$ ($i = 2$), and

$$\begin{aligned} d^2 &= n_3^2 (n_3^2 c^2 / a^2 - k_0^2), \\ b &= k_0^2 - 2 n_3^2 (c^2 + r r_1) / a^2, \\ k_0^2 &= 4 r r_1 / a^2, \quad a^2 = \gamma^2 + (r + r_1)^2, \\ c^2 &= \gamma^2 + r^2. \end{aligned} \tag{9b}$$

Vector Potential:

$$\hat{A}_j(\alpha) = -k_0^2 a / 2r \int_0^u du \begin{cases} 2g^2 \operatorname{sn} u \operatorname{cn} u / (1 - p^2 \operatorname{sn}^2 u) \\ g[1 - 2g^2 \operatorname{sn}^2 u / (1 - p^2 \operatorname{sn}^2 u)] \end{cases}$$
$$(p^2 = n_2^2 / (n_2^2 - 1)^2; j = r, \varphi). \tag{11}$$

Magnetic Field:

$$\hat{H}_l(\alpha) = k_0^2 \gamma / (2ar) \begin{cases} \tilde{H}_r(\alpha) \\ \tilde{H}_\varphi(\alpha) \\ (1/\gamma)[r \tilde{H}_r(\alpha) - r_1 \tilde{H}_z(\alpha)], \end{cases} \quad (l = r, \varphi, z) \tag{12a}$$

where we have defined

$$\tilde{H}_l(\alpha) = \int_0^u du \begin{cases} -P_r + Q_r \operatorname{sn}^2 u + R_r \operatorname{dn}^2 u + S_r \operatorname{nd}^2 u - T_r \operatorname{sd}^2 u - L_r \operatorname{sn}^2 u / (1 - p^2 \operatorname{sn}^2 u) \\ -2 \operatorname{sn} u \operatorname{cn} u [P_\varphi + S_\varphi \operatorname{nd}^2 u + L_\varphi / (1 - p^2 \operatorname{sn}^2 u)] \\ P_z + Q_z \operatorname{sn}^2 u - R_z \operatorname{dn}^2 u - S_z \operatorname{nd}^2 u - T_z \operatorname{sd}^2 u \quad (p^2 = n_2^2 / (n_2^2 - 1); l = r, \varphi, z). \end{cases} \tag{12b}$$

For $\varepsilon \to 0$, the dimensionless parameter $n_1^2 \to k_0^2$ while n_2^2 tends to -2ε and vanishes with it.

The hyper-elliptic integrals (7) and (8) now reduce to the Jacobi form on introducing the following substitutions in the expressions:

$$\begin{aligned} \operatorname{sn}^2 u &= (1 - n_2^2) \sin^2 \alpha / (1 - n_2^2 \sin^2 \alpha), \\ k^2 &= (n_1^2 - n_2^2) / (1 - n_2^2), \\ \operatorname{am} u &= \sin^{-1}[(1 - n_2^2)^{1/2} \sin \alpha / (1 - n_2^2 \sin^2 \alpha)^{1/2}], \end{aligned} \tag{10}$$

where $\operatorname{sn} u$ is a basic Jacobi elliptic function with amplitude $\operatorname{am} u$ and modulus k.

After some tedious but straightforward algebra we get with $g = (1 - n_2^2)^{-1/2}$ the following expressions for the vector potential and magnetic field:

Here $\operatorname{sn} u$, $\operatorname{cn} u$ and $\operatorname{dn} u$ are the three basic Jacobi elliptic functions, and $\operatorname{sd} u$ and $\operatorname{nd} u$ are their quotient and reciprocals in Glaisher's notation [4]. They are given by

$$\operatorname{sd} u = \operatorname{sn} u / \operatorname{dn} u, \quad \operatorname{nd} u = 1 / \operatorname{dn} u.$$

The dimensionless coefficients $P_r, Q_r, \ldots, S_z, T_z$ are expressions involving g, n_1^2, n_2^2 and n_3^2 only and are given below:

$$\begin{aligned} P_r &= 2g \, N_{13}^2 N_{23}^2 / N_{12}^4, & Q_r &= 2g^3 N_{23}^4 / (n_2^2 N_{12}^2), \\ R_r &= g \, N_{23}^4 / N_{12}^4, & S_r &= g \, N_{13}^4 / N_{12}^4, \\ T_r &= 2g^3 N_{13}^4 / (n_1^2 N_{12}^2), & & \\ L_r &= 2g \, L_\varphi = 2g^3 n_3^4 / (n_1^2 n_2^2), & & \end{aligned}$$

$$\begin{aligned} P_\varphi &= g^2 N_{23}^2 [p^2 n_1^2 - n_3^2 (k^2 - p^2)] / (k^2 n_1^2 n_2^2), \\ S_\varphi &= g^2 N_{13}^2 [n_2^2 (k^2 - p^2) - p^2 n_3^2] / (k^2 n_1^2 n_2^2), \\ P_z &= 2g \, n_1^2 n_2^2 / N_{12}^4, \end{aligned}$$

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* The roots (9a) to be real, the square root must be positive definite, i.e. $r^2 \geqq \varepsilon^2 \gamma^2 / (1 - \varepsilon^2)$. Physically it means that the forms (7) and (8) permit their evaluation at all points except at $\bar{r} = (0, 0, z \pm z_1)$. Computations at $\bar{r} = (2\eta\varepsilon, \pi, z)$ with appropriate signs may then be used instead for closed loops.

$$Q_z = n_2^2/n_1^2 \quad T_z = g^3\, n_2^2\, n_3^2/N_{12}^4\,,$$

$$R_z = n_2^4/n_1^4 \quad S_z = g\, n_2^4/N_{12}^4\,, \qquad (12\,c)$$

where for convenience we have defined

$$N_{12}^2 = n_1^2 - n_2^2\,, \quad N_{i3}^2 = n_i^2 - n_3^2 \quad (i = 1, 2).$$

Thus, on integrating (11) and (12 b) we get:

Vector Potential:

$$\hat{A}_j(\theta) = -\frac{k_0^2\, a}{2r}\begin{cases} 2g^2\, I(p^2) & (j = r, \varphi) \quad (14) \\ g\,[(1 - 2/n_2^2)\, K(\theta, k) + 2/n_2^2\, \Pi\,(\theta, p^2, k)]. \end{cases}$$

Magnetic Field:

$$\tilde{H}_l(\theta) = \frac{1}{k^2\, k'^2}\begin{cases} k'^2\,(U_r + k^2\, p^{-2}\, L_r)\, K(\theta, k) + V_r\, E(\theta, k) - W_r\, k^2\, \mathrm{sn}\, u\, \mathrm{cd}\, u - k^2\, k'^2\, p^{-2}\, L_r\, \Pi\,(\theta, p^2, k), \\ 2k'^2\,[P_\varphi\, \mathrm{dn}\, u - S_\varphi\, \mathrm{nd}\, u - k^2\, L_\varphi\, I(p^2)]\,, \\ U_z\, k'^2\, K(\theta, k) - V_z\, E(\theta, k) + W_z\, k^2\, \mathrm{sn}\, u\, \mathrm{cd}\, u \qquad (l = r, \varphi, z). \end{cases} \qquad (15\,a)$$

Final Expressions

Note that for a field point on the arc filament φ equals φ' and the expressions (10) and (11) then become either singular or indeterminate. Physically this is due to the assumed infinitesimal cross-section of the current filament. The corresponding value of α under the angular transformation (A) is $|\alpha| = \pi/2$ and is completely independent of the actual value of either φ or φ', and hence also of φ_1 and φ_2.

Assuming as before for technical reasons, $-\pi \leqq \varphi \leqq \pi$ and $-2\pi \leqq \varphi' \leqq 2\pi$, the angle transformation (A) ensures that $-\pi \leqq \alpha \leqq 2\pi$. Since the integrals of the elliptic functions are defined only on $0 < |\alpha_i| \leqq \pi/2$, we treat the two cases $|\alpha_i| \leqq \pi/2$ and $|\alpha_i| > \pi/2$ separately.

(A) $|\alpha_i| \leqq \pi/2$

Physically this conditions is always satisfied when the field point position vector lies within the azimuthal width of the elliptic arc segment (i.e. when $\varphi_1 \leqq \varphi \leqq \varphi_2$). In this case we define

$$\theta_i = |\alpha_i| \qquad (B)$$

so that $0 < \theta_i \leqq \pi/2$, and

$$\hat{A}_j(|\alpha_i|) = \hat{A}_j(\theta_i) \qquad (j = r, \varphi),$$

$$\hat{H}_l(|\alpha_i|) = \hat{H}_l(\theta_i) \qquad (l = r, \varphi, z). \qquad (13)$$

The integrations in (11) and (12 b) can now be performed with the help of standard recurrence formulas for the integrals of Jacobi elliptic functions [5] and lead to incomplete elliptic integrals $K(\theta, k)$, $E(\theta, k)$ and $\Pi(\theta, p^2, k)$ of the first, second and third kind with molulus k and argument θ.

The dimensionless coefficients $U_r, V_r, \ldots, V_z, W_z$ are functions of the coefficients $P_r, Q_r, \ldots, S_z, T_z$ previously defined in (12 c) and are given by

$$U_m = Q_m + T_m \mp k^2\, P_m\,,$$

$$V_m = W_m + k'^2\,(k^2\, R_m \mp Q_m)\,,$$

$$W_m = k^2\, S_m \mp T_m \quad (m = r, z), \qquad (15\,b)$$

the upper (lower) sign corresponding to $m = r$ $(m = z)$.

The integral $I(p^2)$ is a standard one and is defined by

$$I(p^2) = \int_0^u \frac{\mathrm{d}u\, \mathrm{sn}\, u\, \mathrm{cn}\, u}{(1 - p^2\, \mathrm{sn}^2\, u)}\,.$$

Its value for various cases is already given in [1, (19 b), Part II]. k' is the complementary modulus defined by $k'^2 = 1 - k^2$. The Jacobi elliptic functions $\mathrm{cd}\, u$ and $\mathrm{dc}\, u$ are reciprocals of each other and are designated in Glaisher's notation by

$$\mathrm{cd}\, u = \mathrm{cn}\, u/\mathrm{dn}\, u = 1/\mathrm{dc}\, u\,.$$

(B) $|\alpha_i| > \pi/2$

This condition corresponds to the physical situation in which the field point position vector is exterior to the elliptic arc segment (i.e. when $\varphi_1 > \varphi$ or $\varphi > \varphi_2$).

The ranges of integration in this case are $\pi/2 < |\alpha_i| \leqq \pi$, $\pi < \alpha_i \leqq 3\pi/2$ and $3\pi/2 < \alpha_i \leqq 2\pi$. For the first two we define

$$\theta_i = \pi - |\alpha_i| \qquad (C)$$

while for the last range we set

$$\theta_i = 2\pi - \alpha_i\,, \qquad (D)$$

so that $0 < |\theta_i| \leqq \pi/2$ as before. Separating the ranges of integration at $\pi/2$ and using the symmetry properties of the respective integrands, we again obtain:

Vector Potential:

$$\hat{A}_r(|\alpha_i|) = \hat{A}_r(|\theta_i|)$$

$$\hat{A}_\varphi(|\alpha_i|) = \begin{cases} 2\hat{A}_\varphi(\pi/2) - \text{sgn } \theta_i\, \hat{A}_\varphi(|\theta_i|) \\ \qquad\qquad (\pi/2 < |\alpha_i| \leqq 3\pi/2) \\ 4\hat{A}_\varphi(\pi/2) - \hat{A}_\varphi(\theta_i) \\ \qquad\qquad (3\pi/2 < \alpha_i \leqq 2\pi) \end{cases} \quad (16)$$

Magnetic Field:

$$\hat{H}_\varphi(|\alpha_i|) = \hat{H}_\varphi(|\theta_i|)$$

$$\hat{H}_m(|\alpha_i|) = \begin{cases} 2\hat{H}_m(\pi/2) - \text{sgn } \theta_i\, \hat{H}_m(|\theta_i|) \\ \qquad\qquad (\pi/2 < |\alpha_i| \leqq 3\pi/2) \\ 4\hat{H}_m(\pi/2) - \hat{H}_m(\theta_i) \\ \qquad\qquad (3\pi/2 < \alpha_i \leqq 2\pi) \\ \qquad\qquad\qquad (m = r, z). \end{cases} \quad (17)$$

The functions with argument θ_i in (16) and (17) correspond to those in (14)−(16) while those with argument $\pi/2$ are the same ones obtained by simply replacing θ_i with $\pi/2$. Explicit expressions for the latter are given in the next section.

Note that the expressions (15) and (17) have meaning only as one element of a sum over a continuous set, the sum representing the magnetic induction of a current loop constituting the magnet winding. To obtain physical results we must sum over the complete coil which is made from finite elliptic arc and/or straight segments. The same is also true for the pair (14) and (16).

Examples

Elliptic Loop

When the azimuthal length $(\varphi_2 - \varphi_1)$ of the arc segment equals 2π, we obtain a closed elliptic loop and the expressions for the components of the vector potential and the magnetic field derived above reduce to relatively simple ones.

Symmetry considerations show that in contradistinction to the circular loop case the components $\hat{A}_j$ and $\hat{H}_l$ $(j = r, \varphi; l = r, \varphi, z)$ are φ-dependent. The angle transformation (A) leads to $|\alpha_i| = 1/2\,(\pi \pm \varphi)$, so that for $-\pi \leqq \varphi \leqq \pi$ we obtain $0 \leqq |\alpha_i| \leqq \pi$.

Thus, $0 \leqq |\alpha_i| \leqq \frac{\pi}{2}$ corresponds to the case A while $\pi/2 < |\alpha_i| \leqq \pi$ to the case B, both discussed in the previous section. The symmetry properties for $\varphi = 0$ and π, $\pi/2$ and $3\pi/2$ are then immediately apparent from these expressions. When the field point lies directly on the elliptic loop, they either vanishe or show a singular behaviour.

For the special case $\varphi = 0$, the angle transformation gives $|\alpha_i| = \pi/2$ and the expressions for the vector potential and magnetic field reduce to the following simple ones:

Vector Potential:

$$\hat{A}_j(\pi/2) = -\frac{k_0^2\, a}{2r} \begin{cases} 2g^2\, I(p^2) & (j = r, \varphi), \quad (18) \\ g[(1 - 2/n_2^2)\, K(k) + 2/n_2^2\, \Pi(p^2, k)]. \end{cases}$$

Magnetic Field:

$$\tilde{H}_l(\pi/2) = \frac{1}{k^2\, k'^2} \begin{cases} k'^2(U_r + k^2 p^{-2} L_r)\, K(k) \\ \quad + V_r\, E(k) - k^2 k'^2 p^{-2} L_r\, \Pi(p^2, k) \\ 2k'^2[P_\varphi\, k' - S_\varphi/k' - k^2 L_\varphi\, I(p^2)] \\ U_z\, k'^2\, K(k) - V_z\, E(k) \\ \qquad\qquad (l = r, \varphi, z). \quad (19) \end{cases}$$

$K(k)$, $E(k)$ and $\Pi(p^2, k)$ are complete elliptic integrals of the first, second and third kind with modulus k and argument $\pi/2$, which is suppressed for convenience as usual.

Circular Arc and Loop

When the eccentricity vanishes, the elliptic arc filament and the elliptic loop reduce to a circular arc and to a circular loop of radius r_1, respectively. Since now n_2^2 and n_3^2 both vanish and $n_1^2 = k_0^2$, the corresponding expressions for the vector potential and magnetic field then become

Circular Arc Filament:

$$\hat{A}_j(\alpha) = a/r \begin{cases} \text{dn } u & (j = r, \varphi), \\ (1 - k_0^2/2)\, K(\alpha, k_0) - E(\alpha, k_0) \end{cases} \quad (20\,\text{a})$$

$$\hat{H}_l(\alpha) = 1/ra \begin{cases} \gamma[K(\alpha, k_0) - k_0'^{-2}(1 - k_0^2/2)\, W(\alpha, k_0)] \\ -\gamma\, \text{nd } u & (l = r, \varphi, z), \quad (20\,\text{b}) \\ r\, K(\alpha, k_0) - k_0'^{-2}(r - b k_0^2/2)\, W(\alpha, k_0) \end{cases}$$

where

$$b = r + r_1 \quad \text{and}$$

$$W(\alpha, k) = E(\alpha, k) - k^2\, \text{sn } u\, \text{cd } u.$$

Circular Loop:

$$\hat{A}(\pi/2) = a/r\,[(1 - k_0^2/2)\,K(k_0) - E(k_0)] \qquad (21\,\text{a})$$

$$\hat{H}_m(\pi/2) = 1/a\,r\begin{cases} \gamma\,[K(k_0) - (1 - k_0^2/2)/k_0'^2\,E(k_0)] \\ r\,K(k_0) - (r - b\,k_0^2/2)/k_0'^2\,E(k_0) \end{cases}$$
$$(m = r, z). \qquad (21\,\text{b})$$

Expressions (20 a, b) and (21 a, b) above correspond exactly to those derived directly in Part I [1] for the components of the vector potential and magnetic field of a current carrying circular filament arc and loop, respectively.

Concluding Remarks

The expressions derived in this paper for the components of the vector potential and magnetic field of a plane elliptic arc segment of a current-carrying filament of arbitrary azimuthal length are of comparatively simple algebraic structure and include ony known analytical functions. The complete and incomplete elliptic integrals of the first, second and third kind involved can be computed easily with a high accuracy using the algorithms developed by Bulirsch [6]. Trigonometric representations allow the Jacobi elliptic functions to be calculated exactly.

Since, as in Parts I–IV, the angle transformation we have used always transforms every angle singularity to $\pi/2$, the expressions derived here can be calculated for any elliptic arc filament of arbitrary azimuthal length and for any field point in space with an equal amount of computation work and accuracy. When $|\alpha_i| > \pi/2$, the fact that $|\varphi_1 - \varphi_2| \leqq 2\pi$, i.e. $|\alpha_1 - \alpha_2| \leqq \pi$ may conveniently

θ (deg.)	$\hat{H}_z(\eta\varepsilon, \pi, z_1)$ (1/m)
45	5.40435
40	6.54237
30	8.80767
20	10.76793
10	12.09910
5	12.43053
0	12.56637

Table 1. Computed values of $\hat{H}_z$ at the center of closed elliptic filament loops with minor axis 1 m and various eccentricities generated from a cylinder of radius 0.5 m.

be used while programming to save computation time.

Table 1 shows a number of computed values of $\hat{H}_z$ from (19) at the center of a family of closed elliptic curves generated by lateral sections through a cylinder at angles $0° \leqq \theta \leqq 45°$ with its symmetry axis. They were checked with the filament beam approximation. For small ε the results agree to five significant decimal digits with ten line segments for each computation. The field value for $\theta = 0$ corresponds to a circular filament loop and was also checked directly from (21 b). The computation time-saving is at least by a factor of about ten and increases with the eccentricity. For field points in the immediate neighborhood of the elliptic filament loop the time-saving is greater than a quarter of these values.

In conclusion, we can say that the expressions developed in this paper for calculating the vector potential and magnetic field of a coplanar elliptic arc filament of finite azimuthal length offer, as in preceding papers, a time-saving, highly accurate and efficient computation alternative. They are also in a form that permits a straightforward user-friendly computation algorithm.

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